STUDENTS' UNDERSTANDING OF THE MATHEMATICAL CONCEPTS OF EQUAL AND EQUIVALENCE

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The equivalence relation exhibited as an equals sign is misunderstood by students who consider it to mean 'do something', to indicate the location of the answer and to act as a separator symbol. This presentation reports on a cross-sectional study of students in years P, 2, 4, 6, 8 and 10 to explore understanding of equals and equivalence, conceptions of same and different and knowledge of the properties of equivalence and approaches to the equals sign. Students appeared to predominantly have an operator ('do something') understanding of equals, and to exhibit differing understandings of equals in situations such as 2+3=?, 2+3=5 and x+3=5, of the three properties (reflexivity, symmetry and transitivity) and of the two approaches (static and dynamic).

Mathematics has been categorised by Scandura (1971) as having only three foci: things; relations or relationships between things; and operators, transformations or changes between these things. A relation that is very important in mathematics is equivalence. Equivalence obeys three properties: (a) reflexivity, the relationship relates a thing to itself, i.e. A is related to A; (b) symmetry, if A is related to B, then the opposite direction is also true, i.e. B is related to A; and (c) transitivity, if A is related to B and B is related to C, then A is related to C.

The most used equivalence relation in mathematics is 'equals'. The concept of equality is based on the concepts of 'same' and 'different'. Same and different do not appear to be subject to any major misunderstandings, even by students (e.g. Fischer & Beckey, 1990). However, the equivalence relation exhibited in the equals sign is an elusive concept for students to understand. There is a persistent idea that, rather than expressing a symmetric and transitive relation, the equals sign means 'do something' (Ginsburg, 1982), indicates the location of the answer (Denmark, Barco & Voran, 1976; Kieran, 1992), or is a separator symbol (Kieran, 1992). This misunderstanding appears to continue at the secondary and tertiary levels (e.g. Behr, Erlwanger & Nichols, 1976; Clements, 1982). Clements found that students confused the syntax of the language with the the with semantics of the algebraic sentence, writing 6S=P for there being six students for each professor. In addition, there appears to be lack of attention to the two different ways equality can be approached: (a) in static terms as 'balance', e.g. 2+3 balances 5; and (b) in dynamic terms as 'transformation', e.g. 2+3 changes 2 to 5 by adding 3.

Students' understanding of the equals sign

Behr, Erlwanger & Nichols (1980) studied 6 to 12 year-old
students' understanding of the equals sign in open number sentences. They argued that, to adults, the equals sign in sentences such as 2+4=6 is, intuitively, an abstraction of the notion of sameness and, on a more sophisticated level, an equivalence relation. They further argued that sentences with no plus sign (e.g. 3=3), or more than one plus sign (e.g. 2+1=2+1), do not suggest an action to adults; rather, these sentences are seen to "require a judgement about their truth-value" (Behr, Erlwanger & Nichols, 1980; p.14). They found that students:
(a) understood the equals sign in number sentences such as 2+4= as meaning that something had to be done; (b) did not see 3+2=2+3 in terms of sameness, but rather as an action by restating the sentence as 3+2=5 or 5=2+3; (c) would not accept the equals sign in sentences without it being preceded with one or more operation sign; (d) had "an extreme rigidity" about written sentences and a tendency to perform actions rather than reflect; and (e) did not "change in their thinking about equity as they get older" (p.15). Other researchers (e.g. Ginsburg, 1982; Kieran, 1992; MacGregor, 1991) have also noted that students do not tend to view the equals sign as 'the same as', but rather as an action to be performed (an 'operator'). As Baroody and Ginsburg (1983; p. 199) found:

"... chidren expect written (horizontal) equations to take a particular form: an arithmetic problem consisting of two (or perhaps more) terms of the left, the result on the right, and in between, a connecting ("equals") symbol (e.g. 3+2=5). Children tend to reject equations such as 13=7+6, 6+4=3+7, and 8=8, equations that do not adhere to the typical form and easily lend
themselves to an operator interpretation of equals.".

Research typically indicates that viewing equals as an operator sign persists through primary school (Baroody & Ginsburg, 1983). Moreover, a restricted understanding of equals may continue into secondary and tertiary education and may affect mathematics learning at these levels. For instance, if equals is not viewed as a relational sign, algebra solution strategies (such as adding identical elements to each side of an equation to simplify the expression on one side) may not be meaningful. Another example is MacGregor's (1991) finding that the reversal error commonly made in Clement's (1982) Student-Professor problem, was due to the inappropriate use of the equals sign.

This paper describes a cross-sectional study which intended to expand on the work done by Behr, Erlwanger and Nichols (1976), and explored students' understanding of the mathematical concepts of equals and equivalence in years P, 2, 4, 6, 8, and 10. It explores the extent of students' conceptions about same/difference and the equals sign. It extends these explorations to students' understanding of the properties of equivalence, and knowledge of the static and dynamic approaches to the equals sign.

METHOD

Subjects

The subjects in the study were chosen by their teachers to represent the range of abilities in their classes and included: (a) six students, four girls and two boys, from a private inner-city preschool; (b) eighteen students from an inner-city private catholic primary school, three boys and three girls from each of years 2, 4, and 6; (c) twelve students from an inner-city private catholic secondary school, three girls and three boys from each of years 8 and 10.

Instrument

The instrument used was the "mixed cases" focused clinical interview, an approach to data gathering based on talk aloud as well as traditional interview techniques. Students were interviewed whilst working on the following four sets of equals and equivalence tasks: (a) the notions of same and different - the students were asked to identify same and different for a variety of topic areas; (b) formal understanding of the equals sign - the students were asked what the sign meant in different situations; (c) the equivalence properties - the students' understanding of reflexivity, symmetry, and transitivity were explored; and (d) static and dynamic approaches - the students' meanings of equals as a balancing relationship and as a transformation or change were explored. The tasks involved the students working in situations and contexts appropriate to their year level: counters, blocks, pattern cards and drawings in the early years, and equivalent fractions, similar shapes and number and algebra statements and sentences in the later years. Some novel equality situations were considered in the later years.

Procedure

The students were interviewed at their school in a spare room. The interviews were video-taped. The focus of analysis was the ideas of equals and equivalence held by the students in terms of their present mathematical level.

RESULTS AND DISCUSSION

The video-tapes were transcribed into protocols and analysed for commonalities. From these, categories of understanding that might explain behaviour were inferred.

The students' reactions to the four sets of tasks varied. Some tasks were completed with what appeared to be good understanding, including those relating to same and difference (in accord with Fischer & Beckey, 1990) and to the equivalence property of transitivity. Other tasks were not often completed satisfactorily and appeared to be little understood, particularly those relating to the equivalence property of transitivity and to the transformation approach to equals.

However, behaviours were not always able to be interpreted in a straightforward manner. For instance, although the results with respect to same and difference showed that students could identify same and different, in many cases this only occured under questioning. It showed that, for these students, the idea of considering two things are the same, when this means the same in some attributes while being different in other attributes, in not a natural or easily accepted understanding.

These subtleties in understanding were also evident in meaning of equality. Similar to the literature (Behr, Erlwanger & Nichols, 1980; Ginsburg, 1982; Kieran, 1992; MacGregor, 1991), the equals sign was predominantly seen as an action, e.g. "2+3= leaves something to be done while in 2+3=5 it has been done". This action orientation led to some interesting coclusions from students, e.g. seeing 5=2+3 type examples as being "the wrong way around". However, there were some indications that students see the equals sign as meaning something slightly different in examples such as 2+3=5, 2+3=? and x+3=5. The subtlety in these distinctions appeared to increase with age. For many students, the equals sign in 2+3=5 is seen as the sides being the same while in 2+3=? as a direction to work something out. Older students would often say that the sign means the same thing in all situations but then proceed to assign a different meaning to different situations, e.g. "It is the same, but you have to find the answer in this situation (2+3=?)".

When the equals sign was considered in other sitations, e.g. equivalent fractions, similar shapes, the quality of the students' responses appeared to depend on their familiarity with the context. In the novel situation, used only for the older students, where I+I was made equal to XX for the reason that four popsicle sticks (or four straightlines) were used to make either side, students appeared to have real difficulty coping with having an equals sign in this statement

For the equivalence properties, transitivity appeared to be well understood, symmetry reasonably understood and reflexivity not recognised. When faced with a situation, where a design card or a number is placed in a collection and students are asked to find something in the collection that is the same as that card/number, they would rarely pick up the starting card or number.

Equality relationships were commonly seen in terms of balance and students were skillful in returning ralationships to balance when they were unbalanced. Equality as transformation was less familiar and older students appeared unable to interpret examples such as x+3=5 in terms of change and reversing change, even when an example was worked through with them.

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